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DERIVATION AND ANALYSIS OF A GENERALIZED STANDARD MODEL FOR CONTACT, FRICTION AND WEAR

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Abstract—A model for mechanical contact including friction, wear and heat generation is proposed. By defining an internal state variable for the wear process, a generalized standard model for contact, friction and wear is derived from the principle of virtual power and the fundamental laws of thermodynamics. Within the frame of the generalized standard model some specific constitutive models are presented. For instance, a free energy corresponding to an extension of Signorini's unilateral contact conditions to include the wear process at the interface and having a linear tangential compliance between the relative tangential displacement and the tangential contact traction is suggested. Furthermore, a dual pseudo-potential with a friction and wear limit criterion in agreement with Coulomb's law of friction and Archard's law of wear is given. In order to study existence and uniqueness questions, this pair of free energy and dual pseudo-potential is analysed in a one point elastic quasi-static contact problem with two degrees of freedom and thermal effects neglected. The so-called rate problem is solved.

1. INTRODUCTION

In this paper a continuum thermodynamic model for interfacial phenomena including contact, friction and wear is proposed. The framework is that of small displacements, implying small slip. Consequently, the model is mainly intended for studying fretting, a wear phenomenon arising when contacting surfaces undergo oscillatory displacements with small amplitudes.

Following a line of reasoning developing in works by Onsager (1931), Ziegler (1958; 1963), Coleman and Noll (1963), Moreau (1970; 1974), Halphen and Nguyen (1975), Nguyen (1977), Germain *et al.* (1983) and others, a method for deriving constitutive equations based on the concept of a generalized standard material is used. The method ensures satisfaction of the dissipation inequality by deriving the constitutive equations from a free energy potential and a dissipation potential.

The first step of the method consists in using the method of virtual power and the fundamental principles of thermodynamics, coupled with an internal variable representation of the state, to derive *state laws* and a dissipation inequality. The form of the state laws and the dissipation inequality are specified by the choice of internal variables and the particular form of the free energy.

The next step is to chose evolution laws for the internal variables, so-called *complementary constitutive laws*. From the point of view of the basic principles of thermodynamics there is a lot of freedom in choosing these evolution laws. However, if one chooses to obey a maximum dissipation principle (Onsager, 1931; Ziegler, 1958; 1963), then the evolution laws are expressed by means of gradients of a function of rates of internal variables, a so-called dissipation potential, in the same way as the state laws follow from the free energy. The whole problem of specifying a constitutive law is now reduced to specifying two potentials—the free energy and the dissipation potential. A material obeying such a law is called a *generalized standard material.*

It is important to recognize that due to the work of Moreau (1970; 1974), it is possible to include non-smooth phenomena like plasticity and friction within the class of generalized standard materials, by taking the dissipation potential as convex, but not necessarily differentiable. Such non-differentiable potentials were called pseudo-potentials by Moreau.

Another important observation is that it is admissible to include a dependence on the state in the dissipation potential [see for example Lemaitre and Chaboche (1990)], i.e. a family of potentials are considered. In this way it is possible to treat, within the concept of a generalized standard material, phenomena such as non-associated plasticity and friction with a non-constant normal force. In this respect see also Ziegler (1981).

The above method for deriving constitutive relations was used by Frémond (1987; 1988) to treat material surfaces and to formulate a theory of adhesion. He also extended the above theory in that non-smooth free energies as well as non-smooth dissipation potentials were used. The framework of Frémond was utilized in Klarbring (1990a) to derive different models for frictional contact. In Johansson and Klarbring (1993) these ideas were extended to take into account frictional heat generation and heat transfer across the contact interface. The present paper is a further extension of this line of work where wear is treated in the same spirit.

In this paper, two particular forms of the free energy and one specific dual pseudo-potential are suggested. The two free energies lead to an extension of the classical Signorini conditions of unilateral contact, taking the wear process at the interface into account. The second one also includes a linear tangential compliance between the relative tangential displacement and the tangential contact traction, which is an approximation of the non-linear behaviour observed from experiments (see, e.g. Wriggers *et al.*, 1990).

A dual pseudo-potential with a general friction and wear limit criterion is investigated, from which Coulomb's law of friction and Archard's law of wear are derived. When an elliptic norm is used in the friction and wear criterion, an anisotropic version of Archard's wear law is derived, which is identical to the wear model proposed by Mróz and Stupkiewicz (1994) who assumed that the wear rate is proportional to the rate of the frictional dissipation.

To gain understanding of the constitutive behaviour of the proposed free energies and the dual pseudo-potential corresponding to Coulomb's law of friction and Archard's law of wear, a one point contact problem with two degrees of freedom and thermal effects neglected is studied. The so-called rate problem is solved, i.e. when a contact state is known at a time t the change of the state due to the change of the external loads is determined. Existence and uniqueness of these solutions are discussed. The method of analysis follows Klarbring (1990b) [see also Martins *et al.* (1994)], where conditions for uniqueness and existence of solutions for a one point elastic contact problem with Signorini contact and Coulomb friction were established. Similar conditions are derived for the constitutive model in this paper. These conditions depend on the coefficient of friction, all stiffness coefficients, the contact force, the tangential compliance and the wear parameters.

The contents of this study are as follows: in Section 2 the generalized standard model is derived from the principle of virtual power, the balance of energy and the second law of thermodynamics; in Section 3 constitutive models for the interface, within the frame of the generalized standard model, are suggested and discussed; in Section 4 conditions for uniqueness and existence of solutions to the rate problem are established; and in Section 5 concluding remarks are presented.

2. DERIVATION OF A GENERAL MODEL

Let the open disjoint regions $\Omega^l (l = 1, 2) \subset \mathscr{R}^d (d = 2, 3)$ with piecewise smooth boundaries $\partial\Omega^l$ be occupied by two continuous deformable bodies, see Fig. 1. Both bodies are subjected to body forces $\mathbf{b}$, prescribed tractions $\mathbf{t}^l$ on $\Gamma_t^l \subset \partial\Omega^l$ and fixed displacements on $\Gamma_u^l \subset \partial\Omega^l$. The displacement field of the bodies is denoted by $\mathbf{u}$.

The material boundaries $\Gamma_c^l \subset \partial\Omega^l$ with outward unit normal vectors $\mathbf{n}_c^l$ represent the potential contact surfaces. Since only problems with small displacements will be considered, the potential contact surfaces and the corresponding normal vectors have to be almost identical, i.e. $\Gamma_c^1 \simeq \Gamma_c^2$ and $\mathbf{n}_c^1 \simeq -\mathbf{n}_c^2$. This makes it possible to define a common contact

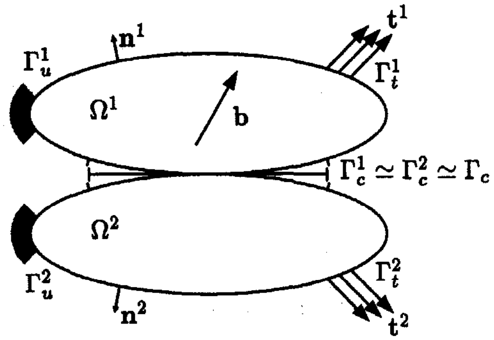


Fig. 1. The two bodies considered, defined by the regions $\Omega^l(l = 1,2) \subset \mathscr{R}^d(d = 2,3)$.

surface $\Gamma_c \simeq \Gamma_c^1 \simeq \Gamma_c^2$ with outward unit normal vector $\mathbf{n}_c \simeq \mathbf{n}_c^1 \simeq -\mathbf{n}_c^2$, i.e. each particle on Γ_c^1 is coupled with a particle on Γ_c^2 in a one-to-one correspondence.

2.1. *The method of virtual power*

The method of virtual power, in the sense of Germain (1973), is used to derive the equilibrium equations and to identify the internal forces as the Cauchy stress and the contact traction vector. For any part $\mathscr{D} \subset \Omega^1 \cup \Omega^2$ such that $\partial\mathscr{D} \cap \Gamma_c^1 \simeq \partial\mathscr{D} \cap \Gamma_c^2$, where $\simeq$ is in the sense indicated above, the virtual power of inertial forces balances the virtual power of all internal and external forces for any virtual velocity field $\hat{\mathbf{v}}$.

Restricting ourselves to quasi-static problems, the principle of virtual power reads, for $\mathscr{D} \subset \Omega^1 \cup \Omega^2$ taken such that $\partial\mathscr{D} \cap \Gamma_c^1 \simeq \partial\mathscr{D} \cap \Gamma_c^2$,

$$\hat{P}_i + \hat{P}_x = 0, \quad \forall \hat{\mathbf{v}} \in \mathscr{V}, \tag{1}$$

where $\mathscr{V}$ is the set of kinematically admissible virtual velocity fields.

The virtual power of internal and external forces are defined as:

$$\hat{P}_i = -\int_{\mathscr{D}} \boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}} \, \mathrm{d}V - \int_{\partial\mathscr{D}\cap\Gamma_c} \mathbf{p} \cdot \hat{\mathbf{w}} \, \mathrm{d}A,$$
$$\hat{P}_x = \int_{\mathscr{D}} \mathbf{b} \cdot \hat{\mathbf{v}} \, \mathrm{d}V + \int_{\partial\mathscr{D}-\Gamma_c} \mathbf{t} \cdot \hat{\mathbf{v}} \, \mathrm{d}A, \tag{2}$$

where $\boldsymbol{\sigma}$ and $\mathbf{p}$ are internal forces in the terminology of the method of virtual power, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\mathbf{v} = \dot{\mathbf{u}}$ is the velocity field and $\mathbf{w} = \mathbf{u}^1 - \mathbf{u}^2$ is the relative displacement vector between coupled particles on Γ_c. A superimposed hat denotes a virtual quantity, a superimposed dot stands for right-hand time derivative, and : and $\cdot$ are the inner products between second-order tensors and vectors, respectively. Notice that all occurring time derivatives in the text are interpreted as right-hand derivatives.

From eqn (1), the equilibrium equations and Cauchy's theorem can be derived. The symmetry of the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{T}}$ implies that only the symmetric part of the internal force $\boldsymbol{\sigma}$ gives a contribution to the virtual power. Therefore, $\boldsymbol{\sigma}$ is considered to be symmetric. With suitable choices of $\hat{\mathbf{v}}$, the following equations are obtained from eqn (1):

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in} \quad \mathscr{D}, \tag{3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{t} \quad \text{on} \quad \partial\mathscr{D} - \Gamma_c, \tag{4}$$

$$\boldsymbol{\sigma}^1 \mathbf{n}_c^1 = -\boldsymbol{\sigma}^2 \mathbf{n}_c^2 = -\mathbf{p} \quad \text{on} \quad \partial\mathscr{D} \cap \Gamma_c, \tag{5}$$

where $\boldsymbol{\sigma}^l$ is the limit of $\boldsymbol{\sigma}$ when approaching Γ_c from within $\mathscr{D} \cap \Omega^l$ and $\mathbf{n}$ is the outward unit normal vector on $\partial\mathscr{D} - \Gamma_c$. The symmetry of $\boldsymbol{\sigma}$, eqns (3) and (4) imply that $\boldsymbol{\sigma}$ can be interpreted as the Cauchy stress and eqn (5) implies that $\mathbf{p}$ can be interpreted as the contact traction vector.

2.2. *The principles of thermodynamics*

In this subsection, the Clausius–Duhem inequalities for the bodies Ω^l and the material interface Γ_c are derived from the first and second laws of thermodynamics by introducing the Helmholtz free energies.

The two basic principles of thermodynamics are postulated as:

$$\dot{\mathscr{E}} = P_x + \mathscr{Q} \quad \forall \mathscr{D},$$

$$\dot{\mathscr{S}} \geqslant \int_{\mathscr{D}} \frac{r}{T} \mathrm{d}V - \int_{\partial\mathscr{D}-\Gamma_c} \frac{\mathbf{q}\cdot\mathbf{n}}{T} \mathrm{d}A \quad \forall \mathscr{D},$$

where $\mathscr{E}$ is the internal energy, P_x is the power of external forces obtained by evaluating eqn (2) for the real velocity, $\mathscr{Q}$ is the heat supply per unit time, $\mathscr{S}$ is the entropy, r is the internal heat production, $\mathbf{q}$ is the heat flux vector and T is the absolute temperature in the bodies Ω^l.

The internal energy $\mathscr{E}$, the entropy $\mathscr{S}$ and the heat supply per unit time $\mathscr{Q}$ are defined as:

$$\mathscr{E} = \int_{\mathscr{D}} \rho e \, \mathrm{d}V + \int_{\partial\mathscr{D}\cap\Gamma_c} E \, \mathrm{d}A,$$

$$\mathscr{S} = \int_{\mathscr{D}} \rho s \, \mathrm{d}V + \int_{\partial\mathscr{D}\cap\Gamma_c} S \, \mathrm{d}A,$$

$$\mathscr{Q} = \int_{\mathscr{D}} r \, \mathrm{d}V - \int_{\partial\mathscr{D}-\Gamma_c} \mathbf{q}\cdot\mathbf{n} \, \mathrm{d}A,$$

where e is the specific internal energy, s is the specific entropy, E is the surface density of internal energy on Γ_c and S the surface density of entropy on Γ_c.

Noting that $\mathscr{D}$ is arbitrary, it is possible to express the first and second laws of thermodynamics on local form as:

$$\left.\begin{aligned} \rho\dot{e} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r - \operatorname{div}\mathbf{q} \\ \rho\dot{s} &\geqslant \frac{r}{T} - \operatorname{div}\left(\frac{\mathbf{q}}{T}\right) \end{aligned}\right\} \quad \text{in} \quad \Omega^1 \cup \Omega^2, \tag{6}$$

$$\left.\begin{aligned} \dot{E} &= \mathbf{p}\cdot\dot{\mathbf{w}} + \mathbf{q}^1\cdot\mathbf{n}_c^1 + \mathbf{q}^2\cdot\mathbf{n}_c^2 \\ \dot{S} &\geqslant \frac{\mathbf{q}^1\cdot\mathbf{n}_c^1}{T^1} + \frac{\mathbf{q}^2\cdot\mathbf{n}_c^2}{T^2} \end{aligned}\right\} \quad \text{on} \quad \Gamma_c, \tag{7}$$

where $\mathbf{q}^l$ and T^l ($l = 1, 2$) are the limits of $\mathbf{q}$ and T, respectively, when approaching Γ_c from within $\mathscr{D} \cap \Omega^l$.

Next, we introduce the Helmholtz free energies ψ, for the volume of bodies, and Ψ, for the area of the contact interface, as:

$$\psi = e - sT, \quad \Psi = E - S\mathscr{T}, \tag{8}$$

where $\mathscr{T}$ is the intrinsic temperature on Γ_c. Moreover, the contact traction vector $\mathbf{p}$ and the relative displacement vector $\mathbf{w}$ are decomposed into a normal component and a tangential vector as:

$$p_N = \mathbf{p}\cdot\mathbf{n}_c, \quad \mathbf{p}_T = (\mathbf{I}-\mathbf{n}_c\otimes\mathbf{n}_c)\mathbf{p}, \quad w_N = \mathbf{w}\cdot\mathbf{n}_c, \quad \mathbf{w}_T = (\mathbf{I}-\mathbf{n}_c\otimes\mathbf{n}_c)\mathbf{w},$$

where the normal vector $\mathbf{n}_c$ was defined previously, $\mathbf{I}$ is the identity tensor and $\otimes$ is the tensor product.

By combining eqns (6)–(8), the Clausius–Duhem inequalities for the bodies Ω^l and the interface Γ_c are obtained as:

$$\rho\dot{\psi} \leqslant \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \rho s\dot{T} - \mathbf{q}\cdot\frac{\nabla T}{T} \quad \text{in} \quad \Omega^1\cup\Omega^2, \tag{9}$$

$$\dot{\Psi} \leqslant p_N\dot{w}_N + \mathbf{p}_T\cdot\dot{\mathbf{w}}_T - S\dot{\mathscr{T}} + \sum_{l=1}^{2}\frac{\mathbf{q}^l\cdot\mathbf{n}_c^l}{T^l}\theta^l \quad \text{on} \quad \Gamma_c, \tag{10}$$

where $\theta^l = T^l - \mathscr{T}$ are the temperature differences between each body Ω^l and the interface Γ_c. From here on, it is assumed that constitutive laws for the bodies Ω^l, satisfying eqn (9), exist and our attention will instead be focussed on the Clausius–Duhem inequality for the interface given in eqn (10).

2.3. *A generalized standard model for the interface*

In this subsection, a generalized standard model for the interface which takes contact, friction, wear and thermal effects into account is derived. The generalized standard model is given by a class of free energies and a class of dual pseudo-potentials from which the state laws and the complementary laws are defined. Using these constitutive laws we obtain a theory where all processes satisfy the reduced dissipation inequality [see, e.g. Lemaitre and Chaboche (1990) and Maugin (1992)].

The modelling of dissipative phenomena, such as friction and wear, may be achieved by the use of internal state variables. We will introduce two internal state variables denoted $\mathbf{w}_T^i$ and w^w.

Firstly, following the ideas of Michalowski and Mróz (1978), Curnier (1984), Cheng and Kikuchi (1985), Klarbring (1990a) and others, the relative tangential displacement is decomposed into one reversible part and one irreversible part:

$$\mathbf{w}_T = \mathbf{w}_T^r + \mathbf{w}_T^i. \tag{11}$$

The reversible part $\mathbf{w}_T^r$, sometimes called the adherence part, is due to the elastic deformations of the asperities, while the irreversible part $\mathbf{w}_T^i$, also called the slipping part, may be attributed partly to the plastic deformations of these asperities but is mainly to the rupture of the junctions between the asperities. This can be compared to the decomposition made in plasticity of the strain into one elastic part and one plastic part.

Secondly, the wear process at the interface is modelled by the internal state variable w^w. Wear is influenced by several interfacial phenomena on a micro-scale, depending on kinematics, material and geometry of the bodies and the environment. These interfacial phenomena are normally explained by four major wear mechanisms [see, e.g. Burwell (1958)], namely, adhesive, abrasive and corrosive wear and surface fatigue, but other minor mechanisms also exist [see, e.g. Suh (1973)]. Eventually, the result of the wear mechanisms can be identified on a macro-scale as wear debris. In our model, the wear is identified as an increase in the gap between the bodies, i.e. the internal state variable w^w is interpreted as a

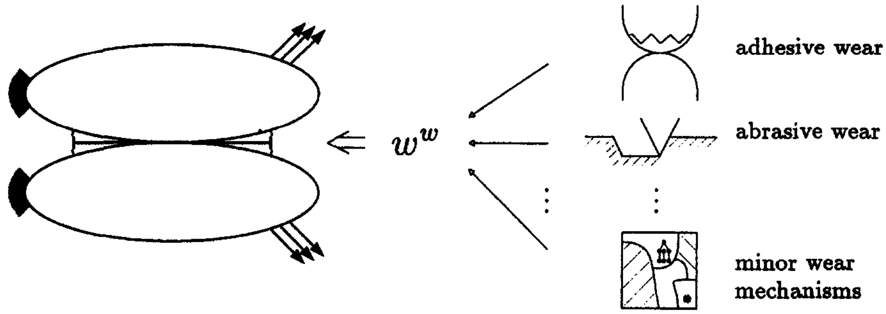


Fig. 2. Interpretation of the internal state variable w^w.

gap in the normal direction $\mathbf{n}_c$ between the bodies owing to the wear mechanisms taking place at the interface, see Fig. 2.

In our contribution Strömberg *et al.* (1995), we introduced one internal state variable for each wear mechanism. However, it turns out that for the particular free energies studied below only the sum of these state variables is of interest. Therefore, this setting is already used here at the outset.

As a general constitutive assumption we consider the following class of free energies:

$$\Psi = \Psi(w_N, \mathbf{w}_T^r, w^w, \mathcal{T}, \theta^1, \theta^2), \tag{12}$$

which is required to be convex with respect to $(w_N, \mathbf{w}_T^r, w^w)$ and differentiable with respect to $(\mathcal{T}, \theta^1, \theta^2)$.

Next, we define the following which will be identified as one part of the state laws:

$$(\tilde{p}_N, \tilde{\mathbf{p}}_T, -\mathcal{W}) \in \partial\Psi(w_N, \mathbf{w}_T^r, w^w, \mathcal{T}, \theta^1, \theta^2) \tag{13}$$

and

$$-\tilde{S} = \frac{\partial \Psi}{\partial \mathcal{T}}, \quad -\Theta^l = \frac{\partial \Psi}{\partial \theta^l} \quad (l = 1, 2). \tag{14}$$

Here $\partial\Psi$ denotes the subdifferential with respect to $(w_N, \mathbf{w}_T^r, w^w)$, holding $(\mathcal{T}, \theta^1, \theta^2)$ fixed. Concerning concepts of convex analysis, such as the subdifferential, see Appendix A or, more fully, e.g. Hiriart-Urruty and Lemaréchal (1993).

The time rate of change of Ψ in eqn (12) at a time t is given by:

$$\begin{aligned}\dot{\Psi} &= \lim_{\Delta t \to 0^+} \frac{\Psi(t+\Delta t) - \Psi(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0^+} \frac{\Psi[w_N(t+\Delta t), \mathbf{w}_T^r(t+\Delta t), w^w(t+\Delta t), \mathcal{T}(t), \theta^1(t), \theta^2(t)] - \Psi(t)}{\Delta t} \\ &\quad + \frac{\partial \Psi}{\partial \mathcal{T}}\dot{\mathcal{T}} + \frac{\partial \Psi}{\partial \theta^1}\dot{\theta}^1 + \frac{\partial \Psi}{\partial \theta^2}\dot{\theta}^2,\end{aligned} \tag{15}$$

where $\Psi(t) = \Psi[w_N(t), \mathbf{w}_T^r(t), w^w(t), \mathcal{T}(t), \theta^1(t), \theta^2(t)]$. A useful expression of eqn (15) is obtained by use of the convexity of Ψ. The definition of the subdifferential in eqn (13) implies that:

$$\begin{aligned}&\Psi[w_N(t+\Delta t), \mathbf{w}_T^r(t+\Delta t), w^w(t+\Delta t), \mathcal{T}(t), \theta^1(t), \theta^2(t)] \geqslant \Psi(t) \\ &\quad + \tilde{p}_N[w_N(t+\Delta t) - w_N(t)] + \tilde{\mathbf{p}}_T \cdot [\mathbf{w}_T^r(t+\Delta t) - \mathbf{w}_T^r(t)] - \mathcal{W}[w^w(t+\Delta t) - w^w(t)],\end{aligned}$$

where $\tilde{p}_{\mathrm{N}}$, $\tilde{\mathbf{p}}_{\mathrm{T}}$ and $\mathscr{W}$ belong to the subdifferential evaluated at time t. By dividing this inequality with a positive time increment Δt and letting Δt approach zero, one obtains from eqn (15) the following inequality:

$$\dot{\Psi} \geqslant \tilde{p}_{\mathrm{N}}\dot{w}_{\mathrm{N}} + \tilde{\mathbf{p}}_{\mathrm{T}} \cdot \dot{\mathbf{w}}_{\mathrm{T}}^{r} - \mathscr{W}\dot{w}^{w} - \tilde{S}\dot{\mathscr{T}} - \sum_{l=1}^{2} \Theta^{l}\dot{\theta}^{l}, \tag{16}$$

where eqn (14) also has been used. This together with the Clausius–Duhem inequality in eqn (10) and the decomposition in eqn (11) give:

$$(p_{\mathrm{N}} - \tilde{p}_{\mathrm{N}})\dot{w}_{\mathrm{N}} + (\mathbf{p}_{\mathrm{T}} - \tilde{\mathbf{p}}_{\mathrm{T}}) \cdot \dot{\mathbf{w}}_{\mathrm{T}} + \tilde{\mathbf{p}}_{\mathrm{T}} \cdot \dot{\mathbf{w}}_{\mathrm{T}}^{i} + \mathscr{W}\dot{w}^{w} - (S - \tilde{S})\dot{\mathscr{T}} + \sum_{l=1}^{2} \frac{\mathbf{q}^{l} \cdot \mathbf{n}_{c}^{l}}{T^{l}}\theta^{l} + \sum_{m=1}^{2} \Theta^{m}\dot{\theta}^{m} \geqslant 0. \tag{17}$$

This inequality must hold for all admissible evolutions of the system. We assume that p_{N}, $\mathbf{p}_{\mathrm{T}}$ and S are state functions, and from the definition in eqn (14) we know that Θ^{m} does not depend on θ^{m} ($m = 1, 2$). Furthermore, if one assumes that $\dot{w}_{\mathrm{N}}$, $\dot{\mathbf{w}}_{\mathrm{T}}$, $\dot{\mathscr{T}}$ and $\dot{\theta}^{m}$ can take arbitrary values in any state, and that the evolution of the internal state variables does not depend on these values, then it follows from eqn (17) that the following must hold:

$$p_{\mathrm{N}} = \tilde{p}_{\mathrm{N}}, \quad \mathbf{p}_{\mathrm{T}} = \tilde{\mathbf{p}}_{\mathrm{T}}, \quad S = \tilde{S}, \tag{18}$$

$$\Theta^{m} = 0\,(m = 1, 2) \Rightarrow \Psi = \Psi(w_{\mathrm{N}}, \mathbf{w}_{\mathrm{T}}^{r}, w^{w}, \mathscr{T}), \tag{19}$$

and

$$\mathbf{p}_{\mathrm{T}} \cdot \dot{\mathbf{w}}_{T}^{i} + \mathscr{W}\dot{w}^{w} + \sum_{l=1}^{2} \frac{\mathbf{q}^{l} \cdot \mathbf{n}_{c}^{l}}{T^{l}}\theta^{l} \geqslant 0. \tag{20}$$

Johansson and Klarbring (1993) considered the case when the admissible values of $\dot{w}_{\mathrm{N}}$ are depending on the state, which in fact is the case for Signorini-like contact conditions, and derived a more general law for p_{N} compared to eqn (18). Also, in the case of rate independent behaviour of friction or plasticity type, the evolution of the internal state variables does actually depend on the rates of the observable variables. Nevertheless, eqns (18)–(20) are sufficient conditions for the inequality in eqn (17) to hold for such cases also, and are therefore assumed to hold in the following.

The state laws are defined by eqns (13), (14) and (18). The left-hand side of the inequality in eqn (20) represents the dissipation at the interface. Furthermore, the associated force $\mathscr{W}$, defined in eqn (13), is identified in eqn (20) as the wear driving force for the wear process.

In order to satisfy the dissipation inequality in eqn (20), we assume that a family of lower semi-continuous convex potentials exists $\Phi = \Phi(\mathbf{p}_{\mathrm{T}}, \mathscr{W}, \theta^{1}, \theta^{2}; \mathscr{P})$, parametrized by $\mathscr{P} = (p_{\mathrm{N}}, \mathbf{w}_{\mathrm{T}}^{r}, \mathbf{w}_{\mathrm{T}}^{i}, w^{w}, \mathscr{T}, T^{1}, T^{2})$, from which the complementary laws are defined by:

$$\left(\dot{\mathbf{w}}_{\mathrm{T}}^{i}, \dot{w}^{w}, \frac{\mathbf{q}^{1} \cdot \mathbf{n}_{c}^{1}}{T^{1}}, \frac{\mathbf{q}^{2} \cdot \mathbf{n}_{c}^{2}}{T^{2}}\right) \in \partial\Phi(\mathbf{p}_{\mathrm{T}}, \mathscr{W}, \theta^{1}, \theta^{2}; \mathscr{P}), \tag{21}$$

and taken such that:

$$0 = \Phi(\mathbf{0}, 0, 0, 0; \mathscr{P}), \quad (\mathbf{0}, 0, 0, 0) \in \partial\Phi(\mathbf{0}, 0, 0, 0; \mathscr{P}). \tag{22}$$

In the terminology of Moreau (1974), $\Phi = \Phi(\mathbf{p}_{\mathrm{T}}, \mathscr{W}, \theta^{1}, \theta^{2}; \mathscr{P})$ is the dual, in the sense of convex analysis, of a pseudo-potential. The dissipation inequality in eqn (20) will always be fulfilled by the complementary laws defined in eqn (21) provided eqn (22) holds, see Appendix A.

The free energy in eqn (19) and the dual pseudo-potential defined above constitute, together with the state laws defined by eqns (13), (14) and (18) and the complementary laws in eqn (21), the generalized standard model for the interface. The model includes contact, friction, wear and thermal effects.

3. CONSTITUTIVE MODELS FOR THE INTERFACE

Signorini's unilateral contact conditions and Coulomb's law of friction are well-known constitutive models for contact and friction. We will extend these laws to take tangential compliance, wear and thermal effects at the contact interface into account. Within the frame of the generalized standard model, two specific free energies and one specific dual pseudo-potential with a general friction and wear limit criterion are proposed.

3.1. *Two specific free energies*

The first specific free energy is an extension of the free energy corresponding to Signorini's unilateral contact conditions, to include wear and thermal effects. The internal variable w^w is used to update the initial gap g between the bodies due to wear processes at the interface. Thus, we consider:

$$\Psi_1 = I_C(w_N, w^w) + I_D(\mathbf{w}_T^r) - \frac{\mathscr{C}}{2\mathscr{T}_0}(\mathscr{T} - \mathscr{T}_0)^2, \tag{23}$$

where

$$C = \{(w_N, w^w) : w_N - w^w - g \leqslant 0\} \quad \text{and} \quad D = \{\mathbf{w}_T^r : \mathbf{w}_T^r = 0\},$$

and I_K denotes the indicator function of a set K, see Appendix A. The closed convex set C corresponds to an extension of Signorini's unilateral contact conditions and the use of the closed convex set D is equivalent to an assumption of zero tangential reversible displacement. The free energy in eqn (23) is also given a thermal dependency: $\mathscr{C}$ is the heat capacity per unit area and $\mathscr{T}_0$ is a reference temperature.

The second form of the free energy which is considered includes tangential compliance at the interface. We modify Ψ_1 so as to read:

$$\Psi_2 = I_C(w_N, w^w) + \tfrac{1}{2}k_T|\mathbf{w}_T^r|^2 - \frac{\mathscr{C}}{2\mathscr{T}_0}(\mathscr{T} - \mathscr{T}_0)^2, \tag{24}$$

where a simple constitutive assumption of a linear elastic behaviour between $\mathbf{p}_T$ and $\mathbf{w}_T^r$ has been added [see, e.g. Wriggers *et al.* (1990)]. k_T is a constant material parameter representing the tangential stiffness of the asperities and $|\cdot|$ is the Euclidean norm. A physically more realistic assumption would be that k_T depends on the contact pressure. This was assumed in Klarbring (1990a).

Let us derive the state laws implied by Ψ_1 and Ψ_2 explicitly, knowing that the subdifferential of an indicator function of a closed convex set is equal to the normal cone of this set, see Appendix A. Inserting eqns (23) and (24) in eqns (13), (14) and (18) gives:

$$\mathscr{W} = p_N \geqslant 0, \quad w_N - w^w - g \leqslant 0, \quad p_N(w_N - w^w - g) = 0, \tag{25}$$

$$\mathbf{p}_T \in \mathscr{R}^2 \quad \text{and} \quad \mathbf{w}_T^r = 0 \quad \text{for} \quad \Psi = \Psi_1, \tag{26}$$

$$\mathbf{p}_T = k_T\mathbf{w}_T^r \Rightarrow \mathbf{w}_T^r = k_T^{-1}\mathbf{p}_T = c_T\mathbf{p}_T \quad \text{for} \quad \Psi = \Psi_2, \tag{27}$$

$$S = \frac{\mathscr{C}}{\mathscr{T}_0}(\mathscr{T} - \mathscr{T}_0). \tag{28}$$

Note that the wear driving force $\mathscr{W}$ is always equal to the contact pressure p_N for these free energies. Also note that Ψ_1 and Ψ_2 only differ in that Ψ_2 implies a tangential compliance at the contact interface, see eqns (26) and (27).

The change in internal energy eqn (7) can be expressed by use of these state laws. Equations (7) and (8) and (23)–(28) give in case of Ψ_1 that:

$$\dot{E} = \frac{\mathscr{C}}{\mathscr{T}_0}\mathscr{T}\dot{\mathscr{T}} = p_N \dot{w}^w + \mathbf{p}_T \cdot \dot{\mathbf{w}}_T^i + \sum_{l=1}^{2} \mathbf{q}^l \cdot \mathbf{n}_c^l, \tag{29}$$

and in case of Ψ_2 that:

$$\dot{E} = \frac{\mathscr{C}}{\mathscr{T}_0}\mathscr{T}\dot{\mathscr{T}} + k_T \mathbf{w}_T^r \cdot \dot{\mathbf{w}}_T^r = p_N \dot{w}^w + k_T \mathbf{w}_T^r \cdot \dot{\mathbf{w}}_T^r + k_T \mathbf{w}_T^r \cdot \dot{\mathbf{w}}_T^i + \sum_{l=1}^{2} \mathbf{q}^l \cdot \mathbf{n}_c^l. \tag{30}$$

Obviously, eqns (29) and (30) serve as evolution laws for the intrinsic temperature at the interface.

3.2. *A specific dual pseudo-potential*

A dual pseudo-potential with a general friction and wear limit criterion, and a thermal dependency similar to Fourier's heat diffusion law for solid bodies are suggested and investigated. The proposal is:

$$\Phi = I_{F(\mathscr{P})}(\mathbf{p}_T, \mathscr{W}) + \frac{1}{2}\sum_{l=1}^{2} \frac{\vartheta^l}{T^l}(\theta^l)^2, \tag{31}$$

where

$$F(\mathscr{P}) = \{(\mathbf{p}_T, \mathscr{W}) : \mathscr{F}(\mathbf{p}_T, \mathscr{W}; \mathscr{P}) \leq 0\}$$

is a closed convex set, $\mathscr{F}(\mathbf{p}_T, \mathscr{W}; \mathscr{P})$ is a quasi-convex function describing the friction and wear limit criterion, as well as the sliding rule and the wear law, and $\vartheta^l = \vartheta^l(\mathscr{P})$ is thermal contact conductances. Concerning more explicit relations for thermal contact conductances [see, e.g. Fried (1969)].

The complementary laws (21) are expressed with eqn (31) as:

$$(\dot{\mathbf{w}}_T^i, \dot{w}^w) \in N_{F(\mathscr{P})}(\mathbf{p}_T, \mathscr{W}), \tag{32}$$

$$\mathbf{q}^l \cdot \mathbf{n}_c^l = \vartheta^l \theta^l \; (l = 1, 2), \tag{33}$$

where $N_{F(\mathscr{P})}$ denotes the normal cone of the set $F(\mathscr{P})$. Here eqn (32) defines the friction and wear laws, and (33) is the equation governing the heat flow across the contact interface. If $\mathscr{F}(\mathbf{p}_T, \mathscr{W}; \mathscr{P})$ is differentiable with respect to $\mathbf{p}_T$ and $\mathscr{W}$, then we obtain from eqn (32):

$$\left.\begin{aligned} \dot{\mathbf{w}}_T^i &= \dot{\lambda}\frac{\partial \mathscr{F}}{\partial \mathbf{p}_T} \\ \dot{w}^w &= \dot{\lambda}\frac{\partial \mathscr{F}}{\partial \mathscr{W}} \end{aligned}\right\} \dot{\lambda} \geq 0, \quad \mathscr{F}(\mathbf{p}_T, \mathscr{W}; \mathscr{P}) \leq 0, \quad \dot{\lambda}\mathscr{F}(\mathbf{p}_T, \mathscr{W}; \mathscr{P}) = 0. \tag{34}$$

Several choices of $\mathscr{F}(\mathbf{p}_T, \mathscr{W}; \mathscr{P})$ are possible. A simple constitutive assumption of a friction and wear model is an extension of Coulomb's friction cone. Let:

$$\mathscr{F}(\mathbf{p}_{\mathrm{T}}, \mathscr{W}; \mathscr{P}) = |\mathbf{p}_{\mathrm{T}}| - \mu p_{\mathrm{N}} + k p_{\mathrm{N}} \mathscr{W}, \tag{35}$$

where k is a wear coefficient, then the wear law in eqn (34) becomes:

$$\dot{w}^{w} = k p_{\mathrm{N}} |\dot{\mathbf{w}}_{\mathrm{T}}^{i}|. \tag{36}$$

Thus, the wear rate is proportional to the sliding velocity and the contact pressure, which is in agreement with experiments [see, e.g. Rabinowicz (1965)]. A similar friction and wear limit criterion as in eqn (35) was suggested by Curnier (1984), with $k p_{\mathrm{N}} \mathscr{W}$ replaced with a force of wear associated to a cumulated slip.

By choosing $k = k_a/3p_s$ where k_a is a wear constant that Archard (1953) interpreted as the probability that a fragment will be formed at an adhesive joint and p_s is the penetration hardness of the softer material, we achieve a local form of Archard's wear law† from eqn (36), i.e.

$$\dot{w}^{w} = \frac{k_a p_{\mathrm{N}} |\dot{\mathbf{w}}_{\mathrm{T}}|}{3 p_s}. \tag{37}$$

As we have already seen for Ψ_1 in eqn (23) and Ψ_2 in eqn (24), the wear driving force $\mathscr{W}$ is equal to the normal contact pressure p_{N} for some specific classes of free energies. In such cases the softening of the Coulomb friction criterion induced by using eqn (35) can be removed by taking:

$$\mathscr{F}(\mathbf{p}_{\mathrm{T}}, \mathscr{W}; \mathscr{P}) = |\mathbf{p}_{\mathrm{T}}| - \mu p_{\mathrm{N}} - \frac{k_a p_{\mathrm{N}}^2}{3 p_s} + \frac{k_a p_{\mathrm{N}} \mathscr{W}}{3 p_s}, \tag{38}$$

which does not affect the form of Archard's law of wear given in eqn (37). The friction and wear criterion in eqn (35) is discussed in Section 3.3., and the criterion in eqn (38) is analysed in Section 4.

Let us change the Euclidean norm used in eqns (35) and (38) to an elliptic norm. Let the rectangular coordinate system x and y specify the tangential plane normal to $\mathbf{n}_c$, and let $p_{\mathrm{T}X}$ and $p_{\mathrm{T}Y}$ be the components of $\mathbf{p}_{\mathrm{T}}$ in this coordinate system. Then a change of the Euclidean norm in eqns (35) and (38) to:

$$\|\mathbf{p}_{\mathrm{T}}\| = \sqrt{\left(\frac{p_{\mathrm{T}X}}{\alpha_X}\right)^2 + \left(\frac{p_{\mathrm{T}Y}}{\alpha_Y}\right)^2}, \quad 0 < \frac{\alpha_X}{\alpha_Y} \leqslant 1 \tag{39}$$

takes anisotropic effects of the surfaces into consideration. The principle axes of the ellipse (α_X, α_Y) account for the existence of preferred directions of slip at the contact interface. Anisotropic friction conditions have been studied by Curnier (1984), Michalowski and Mróz (1978), He and Curnier (1993), and others.

With the elliptic norm (39) in eqns (35) and (38), we obtain an anisotropic version of Archard's wear law from eqn (34), i.e.

$$\dot{w}^{w} = \frac{k_a p_{\mathrm{N}} \sqrt{(\alpha_X \dot{w}_{\mathrm{T}X}^{i})^2 + (\alpha_Y \dot{w}_{\mathrm{T}Y}^{i})^2}}{3 p_s}. \tag{40}$$

Thus, the wear rate will vary with the orientation of the sliding velocity vector. Such correlations between friction and wear have been reported by Jacobs *et al.* (1990), Miyoshi and Buckley (1982), and others. Moreover, an elliptic norm is physically reasonable when considering wear processes, as wear may induce anisotropic roughness at the surfaces. One

† In our contribution (Strömberg *et al.*, 1995), Coulomb's friction cone was extended with $k_a \mathscr{W}^2/6p_s$ leading to Archard's wear law when $\mathscr{W} = p_{\mathrm{N}} > 0$. Unfortunately, this extension implies that the wear rate can take arbitrary values when $p_{\mathrm{N}} = 0$, which is not in agreement with physical intuition.

may think of a situation when α_X and α_Y depend on some cumulated wear in each of the principle directions x and y of the interface.

Anisotropic wear models were investigated by Mróz and Stupkiewicz (1994). They assumed that the wear rate is proportional to the frictional dissipation. With our notations this reads for the elliptic norm (39)

$$\dot{w}^w = k_M \mathbf{p}_\mathrm{T} \cdot \dot{\mathbf{w}}_\mathrm{T}^i = k_M \sqrt{(\alpha_X \dot{w}_{\mathrm{T}X}^i)^2 + (\alpha_Y \dot{w}_{\mathrm{T}Y}^i)^2}\, \|\mathbf{p}_\mathrm{T}\|, \tag{41}$$

where k_M is a wear constant. This law is similar to the wear law derived in eqn (40) as $\|\mathbf{p}_\mathrm{T}\|$ is closely related to p_N. For instance, if one assumes that $\|\mathbf{p}_\mathrm{T}\| = \mu p_\mathrm{N}$ when friction is developed, then eqn (40) can be obtained from eqn (41) by replacing k_M with $k_a/3p_s\mu$.

However, in our model the wear process is a part of the dissipation and the change in internal energy. For the friction and wear limit criterion in eqn (38) corresponding to Coulomb's law of friction and Archard's law of wear, the dissipation inequality (20) becomes with $\mathscr{W} = p_\mathrm{N}$:

$$\mu p_\mathrm{N} |\dot{\mathbf{w}}_\mathrm{T}^i| + \frac{k_a p_\mathrm{N}^2 |\dot{\mathbf{w}}_\mathrm{T}^i|}{3p_s} + \sum_{l=1}^{2} \frac{\vartheta^l (\theta^l)^2}{T^l} \geqslant 0. \tag{42}$$

Here the relations in eqn (33) have also been used. The change in internal energy for Ψ_1 in eqn (29) can be reformulated to:

$$\dot{E} = \frac{\mathscr{C}}{\mathscr{T}_0} \mathscr{T} \dot{\mathscr{T}} = \frac{k_a p_\mathrm{N}^2 |\dot{\mathbf{w}}_\mathrm{T}^i|}{3p_s} + \mu p_\mathrm{N} |\dot{\mathbf{w}}_\mathrm{T}^i| + \sum_{l=1}^{2} \vartheta^l \theta^l. \tag{43}$$

It can be seen that the change in internal energy is almost identical to the dissipation in eqn (42). The change in internal energy for Ψ_2 in eqn (30) is almost identical to eqn (43) except for the additional term $k_\mathrm{T} \mathbf{w}_\mathrm{T}^r \cdot \dot{\mathbf{w}}_\mathrm{T}^r$.

3.3. *Discussion of the friction criteria*

By choosing the function $\mathscr{F}(\mathbf{p}_\mathrm{T}, \mathscr{W}; \mathscr{P})$ as in eqn (35) we obtain Archard's law of wear (37), [see also Holm (1946)], and the friction law:

$$\dot{\mathbf{w}}_\mathrm{T}^i = \dot{\lambda} \frac{\mathbf{p}_\mathrm{T}}{|\mathbf{p}_\mathrm{T}|}, \quad \dot{\lambda} \geqslant 0, \quad |\mathbf{p}_\mathrm{T}| - \mu p_\mathrm{N} + \frac{k_a p_\mathrm{N} \mathscr{W}}{3p_s} \leqslant 0, \quad \dot{\lambda}\left(|\mathbf{p}_\mathrm{T}| - \mu p_\mathrm{N} + \frac{k_a p_\mathrm{N} \mathscr{W}}{3p_s}\right) = 0, \tag{44}$$

which is a slight modification of Coulomb's law of friction. Note that $\mathscr{W} = p_\mathrm{N}$ for the particular choices of the Helmholtz free energy of the contact interface made in this paper.

By modifying $\mathscr{F}(\mathbf{p}_\mathrm{T}, \mathscr{W}; \mathscr{P})$ to the form in eqn (38) it is possible to obtain Archard's law of wear and Coulomb's law of friction, within the presented thermodynamical framework. This is tempting since these are accepted first approximations of wear and friction models, which have stood the test of time. We believe, however, that there are reasons to tentatively retain $\mathscr{F}(\mathbf{p}_\mathrm{T}, \mathscr{W}; \mathscr{P})$ in the form of eqn (35), and thus to consider the friction law (44). These reasons are in short:

1. Taking $\mathscr{F}(\mathbf{p}_\mathrm{T}, \mathscr{W}; \mathscr{P})$ in the form of eqn (35) is the simplest extension of Coulomb's friction criteria we have found that serves the purpose of incorporating Archard's law of wear within our thermodynamical framework. The resulting friction law differs from Coulomb's law, but for practical purposes, with $k_a \ll \mu$, the difference is unimportant.
2. Nevertheless, taking $\mathscr{F}(\mathbf{p}_\mathrm{T}, \mathscr{W}; \mathscr{P})$ in the form of eqn (35) does introduce a difference from the classical Coulomb's law. However, we believe that the corresponding friction law (44) is acceptable from an experimental and intuitive point of view.

To expand on point 2 above, we note that if $\mathscr{W} = p_N$, which is obtained for the choices of Ψ considered in this paper, we obtain the friction criterion:

$$|\mathbf{p}_T| \leqslant \mu p_N - \frac{k_a p_N^2}{3p_s}. \tag{45}$$

The validity of Coulomb's criteria $|\mathbf{p}_T| \leqslant \mu p_N$ as a first approximation is well verified experimentally, but experiments also show a dependency of the friction coefficient on a number of parameters, and a decrease of the friction coefficient with increasing pressure, as predicted by eqn (45), is sometimes observed [see, e.g. Suh (1982)].

To defend eqn (45) from an intuitive point of view, we note that Archard's wear law has been interpreted using an adhesive model involving formation and breaking of adhesive joints between asperities [see Archard, (1953); Rabinowicz, (1965)]. This is shown in Fig. 3. In this wear model the wear constant k_a is interpreted as the probability that an adhesive joint will break somewhere else than where it was formed, and later break along the path where it was formed, i.e. a loose wear particle is formed. Rabinowicz (1965) explains the second stage in the formation of wear fragments with a model involving elastic strain energy in the fragment.

If we assume that the reason that a joint breaks along an alternative path is that a path exists where the joint is weaker than along the path where it was formed, it is reasonable to expect a reduction in the friction force proportional to k_a. This is the behaviour predicted by eqn (45).

To further exploit the adhesive model we note that since, obviously, $|\mathbf{p}_T| \geqslant 0$, it is necessary to impose the following condition on eqn (45):

$$p_N \leqslant \frac{3p_s\mu}{k_a}. \tag{46}$$

In most applications we have $k_a \ll \mu$, in which case this condition is fulfilled. However, interpreting k_a as probability, it could, at worst, be equal to 1. Further, in an adhesive friction model, the coefficient of friction equals the ratio between the shear flow stress and the normal flow pressure, i.e. $\mu = \tau_s/p_s$. A simple assumption is to put $\tau_s = \sigma_y/2$ and $p_s = 3\sigma_y$, where σ_y is the plastic yield strength in uniaxial compression, to give $\mu = 1/6$. More realistic assumptions, however, must recognize the fact that the normal flow pressure is reduced when tangential tractions are present in addition to the normal pressure, due to the coupling between the stress components in a plastic yield criterion. If we put $k_a = 1$ and $\mu = 1/3$ in eqn (46) we obtain:

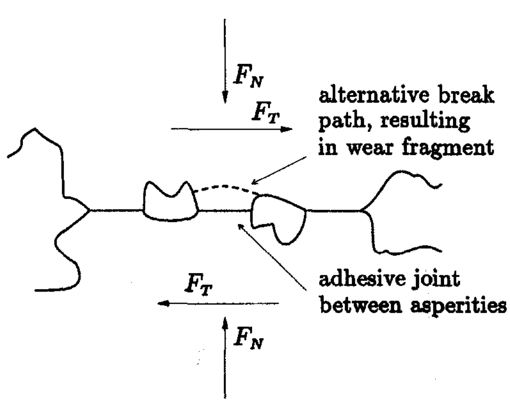


Fig. 3. Two interacting surface asperities.

$$p_N \leqslant p_s,$$

which is a condition that must be made in an adhesive friction model anyway, as $p_N = p_s$ when the true contact area becomes equal to the apparent area. Thus, it is seen that the condition $|\mathbf{p}_T| \geqslant 0$, which must be imposed on eqn (45), fits into an interpretation in terms of an adhesive friction model.

4. A ONE POINT ELASTIC CONTACT PROBLEM

In this section, a contact problem involving one contact point with two degrees of freedom and thermal effects neglected is considered. The so-called rate problem is solved, i.e. for a given state of contact and a given rate of change of the external loading, the rate of change of the state is determined. This problem is solved for the free energy (24) corresponding to Signorini contact with tangential compliance and the dual pseudo-potential (31) with the friction and wear limit criterion (38) equivalent to Coulomb's law of friction and Archard's law of wear. Existence and uniqueness of the solutions are discussed. The same problem for Signorini contact with Coulomb friction was considered by Klarbring (1990a).

4.1. *The model, the state laws and the complementary laws*

Let us consider a class of one point contact problems where the external forces (F_T, F_N), the contact forces (P_T, P_N) and the relative displacements (w_T, w_N), see Fig. 4, are related by:

$$\begin{Bmatrix} F_T \\ F_N \end{Bmatrix} - \begin{Bmatrix} P_T \\ P_N \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{Bmatrix} w_T \\ w_N \end{Bmatrix}, \tag{47}$$

where the matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

is positive definite.

As only one point is considered the contact forces above take the place of the tractions used before. Moreover, with two degrees of freedom, thermal effects neglected and the initial gap g taken to be equal to zero, the state laws corresponding to Ψ_2 in eqns (25) and (27) can be written as:

$$P_N \geqslant 0, \quad w_N - w^w \leqslant 0, \quad P_N(w_N - w^w) = 0, \quad w_T^e = C_T P_T \tag{48}$$

and the complementary laws in eqn (32) corresponding to $\mathscr{F}(P_T, \mathscr{W}; \mathscr{P})$ in eqn (38) can be written as:

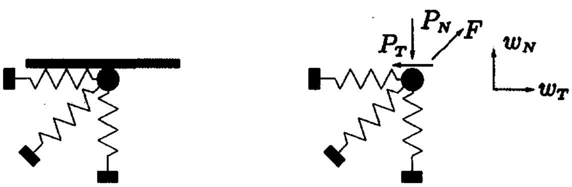


Fig. 4. One point elastic contact problem.

$$\dot{w}_T^i = \dot{\lambda}\frac{P_T}{|P_T|}, \quad \dot{w}^w = \dot{\lambda}\frac{k_a P_N}{3P_s}, \quad \dot{\lambda} \geqslant 0, \quad |P_T| - \mu P_N \leqslant 0, \quad \dot{\lambda}(|P_T| - \mu P_N) = 0 \tag{49}$$

if $P_N > 0$, and:

$$\dot{w}_T^i \in \mathscr{R}, \quad \dot{w}^w = 0 \quad \text{if} \quad P_N = 0.$$

Here c_T and p_s have been replaced by C_T and P_s.

In the following analysis, the result for Signorini contact without reversible tangential displacement is simply achieved by letting $C_T \to 0$, and the result for Coulomb friction without wear by letting $k_a \to 0$.

4.2. *The contact states and the rate laws*

The state of contact is given by $(w_N, w^w, w_T, P_N, P_T)$. In each particular state of contact, conditions on the rate of change of the state are determined by the so-called rate laws. These can be derived by making a Taylor expansion in time of the state laws in eqn (48) and the complementary laws in eqn (49), and then evaluate these expansions for each particular contact state. The Taylor expansions of eqns (48) and (49) can be found in Appendix B. For each particular state, the following rate laws are derived:

a. If $P_T = P_N = 0$ and $w_N - w^w < 0$, then

$$\dot{P}_N = \dot{P}_T = 0, \quad \dot{w}_N \in \mathscr{R}, \quad \dot{w}_T^i \in \mathscr{R}, \quad \dot{w}^w = w_T^r = \dot{w}_T^r = 0.$$

b. If $P_T = P_N = 0$ and $w_N - w^w = 0$, then

$$\dot{P}_N \geqslant 0, \quad \dot{w}_N \leqslant 0, \quad \dot{P}_N \dot{w}_N = 0, \quad \dot{w}^w = w_T^r = 0, \quad \dot{w}_T^r = C_T \dot{P}_T,$$

$$|\dot{P}_T| \leqslant \mu \dot{P}_N, \quad \dot{P}_T = \mu \dot{P}_N \operatorname{sgn}(\dot{w}_T^i) \text{ if } \dot{w}_T^i \neq 0.$$

c. If $|P_T| < \mu P_N$, $P_N > 0$ and $w_N - w^w = 0$, then

$$\dot{P}_N \in \mathscr{R}, \quad \dot{P}_T \in \mathscr{R}, \quad \dot{w}_N = \dot{w}^w = \dot{w}_T^i = 0, \quad w_T^r = C_T P_T, \quad \dot{w}_T^r = C_T \dot{P}_T.$$

d. If $|P_T| = \mu P_N$, $P_N > 0$ and $w_N - w^w = 0$, then

$$\dot{P}_N \in \mathscr{R}, \quad \dot{P}_T \operatorname{sgn}(P_T) \leqslant \mu \dot{P}_N, \quad \dot{P}_T \operatorname{sgn}(P_T) = \mu \dot{P}_N \quad \text{if} \quad \dot{w}_T^i \neq 0,$$

$$\dot{w}_N = \dot{w}^w = \frac{k_a P_N |\dot{w}_T^i|}{3P_s}, \quad \dot{w}_T^i = |\dot{w}_T^i| \operatorname{sgn}(P_T), \quad w_T^r = C_T P_T, \quad \dot{w}_T^r = C_T \dot{P}_T.$$

4.3. *The rate problem*

For a given state and change in external loading, the rate problem is to find the rate of change of state. This can be found by using the rate form of eqn (47):

$$\begin{Bmatrix} \dot{F}_T \\ \dot{F}_N \end{Bmatrix} - \begin{Bmatrix} \dot{P}_T \\ \dot{P}_N \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{Bmatrix} \dot{w}_T \\ \dot{w}_N \end{Bmatrix}, \tag{50}$$

together with the rate laws given in Section 4.2. For each contact state, a–d, the rate of change of state depending on the rate of change in external loading are obtained and summarized below together with comments and illustrations.

a. The rate of change of the state is (with $\dot{P}_N = \dot{P}_T = \dot{w}_T^r = 0$)

$$\begin{Bmatrix} \dot{w}_T \\ \dot{w}_N \end{Bmatrix} = \frac{1}{\det[A]} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{bmatrix} \begin{Bmatrix} \dot{F}_T \\ \dot{F}_N \end{Bmatrix}.$$

b. Three different types of solutions are found—separation, stick and slip. The conditions on the rates of the external forces are for:

i. separation (i.e. $\dot{w}_N < 0$, $\dot{P}_N = \dot{P}_T = \dot{w}_T^r = 0$):

$$a_{11}\dot{F}_N - a_{12}\dot{F}_T < 0,$$

ii. stick (i.e. $\dot{w}_N = 0$, $\dot{P}_N \geqslant 0$, $\dot{w}_T^i = 0$ and $\dot{w}_T^r = C_T\dot{P}_T$):

$$a_{12}C_T\dot{F}_T \leqslant (1+a_{11}C_T)\dot{F}_N, \quad |\dot{F}_T| + \mu a_{12}C_T\dot{F}_T \leqslant \mu(1+a_{11}C_T)\dot{F}_N,$$

iii. slip (i.e. $\dot{w}_N = \dot{w}^w = 0$, $\dot{P}_N \geqslant 0$, $\dot{w}_T^i \neq 0$ and $\dot{w}_T^r = C_T\dot{P}_T$):

$$\dot{w}_T^i \operatorname{sgn}(\dot{w}_T^i) = \frac{(\operatorname{sgn}(\dot{w}_T^i) + \mu a_{12}C_T)\dot{F}_T - \mu(1+a_{11}C_T)\dot{F}_N}{a_{11} - \mu a_{12}\operatorname{sgn}(\dot{w}_T^i)} > 0,$$

$$\dot{P}_N = \frac{a_{11}\dot{F}_N - a_{12}\dot{F}_T}{a_{11} - \mu a_{12}\operatorname{sgn}(\dot{w}_T^i)} \geqslant 0.$$

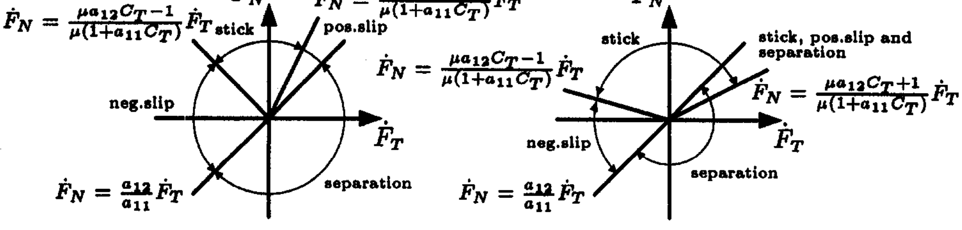
In this contact state there exist solutions for all loading directions but they may be non-unique, which can be seen if one compares the inequalities in b.i–b.iii, defining the solution to be separation, stick and/or slip. The uniqueness depends on the sign of the denominator in b.iii, i.e. $a_{11} - \mu a_{12}\operatorname{sgn}(\dot{w}_T^i)$.

If $a_{11} > \mu|a_{12}|$, then the denominator is greater than zero for both positive and negative slip, and unique solutions exist for all loading directions. This is illustrated in Fig. 5(a) for the case when $a_{12} > 0$. On the other hand, if $a_{11} < \mu|a_{12}|$, then the denominator is negative for positive slip (or negative slip) when $a_{12} > 0$ (or $a_{12} < 0$), and non-unique solutions may appear. This is shown in Fig. 5(b) for the case when $a_{12} > 0$. Finally, if $a_{11} = \mu|a_{12}|$, then the denominator is equal to zero for positive or negative slip, depending on the sign of a_{12}, and the numerators in b.iii must then be equal to zero. In such cases, one finds non-unique solutions identified as stick and slip for loading directions defined by the first and second numerators in b.iii.

In conclusion, it exists solutions for all loading directions, but for:

$$\mu \geqslant \frac{a_{11}}{|a_{12}|},$$

non-uniqueness of solutions may appear. Furthermore, the value of C_T has no influence on



(a) (b)

Fig. 5. Intermediate contact state (b). Different solutions depending on loading directions when $a_{12} > 0$: (a) $\mu < a_{11}/a_{12}$; (b) $\mu > a_{11}/a_{12}$.

this condition for existence and uniqueness of solutions, but it affects the size of the domains of stick and slip solutions. In the limit $C_T \to 0$, we obtain the domain of solutions for Signorini contact with Coulomb friction.

c. In this contact state unique solutions exist for all loading directions. The solutions are (with $\dot{w}_N = \dot{w}_T^i = 0$ and $\dot{w}_T^r = C_T \dot{P}_T$)

$$\dot{P}_T = \frac{\dot{F}_T}{1+a_{11}C_T}, \quad \dot{P}_N = \frac{(1+a_{11}C_T)\dot{F}_N - a_{12}C_T\dot{F}_T}{1+a_{11}C_T}.$$

Although there is no slip and consequently no wear in this contact state, there can still be a change in the relative tangential displacement between the bodies. In fretting situations, a threshold has been observed on the amplitude of the tangential displacement below which no wear is developed (see e.g. Waterhouse, 1984). Thus this wear model is in agreement with such observations, if one assumes that the threshold can be explained by the elastic response of the asperities.

d. Two different types of solutions are identified—stick and slip. We get the following conditions on the rates of the external forces for:

i. stick (i.e. $\dot{w}_T^i = 0$, $\dot{w}_N = 0$ and $\dot{w}_T^r = C_T \dot{P}_T$):

$$[\operatorname{sgn}(P_T) + \mu a_{12} C_T]\dot{F}_T \leqslant \mu(1+a_{11}C_T)\dot{F}_N,$$

ii. slip (i.e. $\dot{w}_T^i \neq 0$, $\dot{w}_N = \dfrac{k_a P_N |\dot{w}_T^i|}{3P_s}$ and $\dot{w}_T^r = C_T \dot{P}_T$):

$$[\operatorname{sgn}(P_T) + \mu a_{12} C_T]\dot{F}_T > \mu(1+a_{11}C_T)\dot{F}_N \quad \text{if} \quad \Lambda > 0,$$

$$[\operatorname{sgn}(P_T) + \mu a_{12} C_T]\dot{F}_T = \mu(1+a_{11}C_T)\dot{F}_N \quad \text{if} \quad \Lambda = 0,$$

$$[\operatorname{sgn}(P_T) + \mu a_{12} C_T]\dot{F}_T < \mu(1+a_{11}C_T)\dot{F}_N \quad \text{if} \quad \Lambda < 0,$$

where

$$\Lambda = a_{11} - \mu a_{12} \operatorname{sgn}(P_T) + \frac{k_a P_N}{3P_s}(a_{12} \operatorname{sgn}(P_T) - \mu a_{22}) - \mu C_T \frac{k_a P_N}{3P_s}(a_{11}a_{22} - a_{12}^2). \quad (51)$$

In this contact state existence and uniqueness of solutions are depending on the sign of Λ in eqn (51). This can be seen if one compares the inequalities in d.i and d.ii, defining the solution to be stick and/or slip. If $\Lambda > 0$, then there exist unique solutions for all loading directions. Otherwise, i.e. $\Lambda \leqslant 0$, there are two solutions, one solution or no solution for different loading directions.

The expressions of Λ in eqn (51) depend on several parameters. Let us study this expression for some special cases by letting k_a and C_T approach zero.

Firstly, if $C_T \neq 0$ and $k_a \to 0$, then we get the same condition on uniqueness and existence of solutions as for Signorini contact with Coulomb friction, i.e. eqn (51) becomes:

$$\Lambda_C = a_{11} - \mu a_{12} \operatorname{sgn}(P_T).$$

For sufficiently small friction coefficients Λ_C is greater than zero and unique solutions exist for all loading directions. On the other hand, if

$$\mu \geqslant \frac{a_{11}}{|a_{12}|},$$

then non-uniqueness and non-existence of solutions appear for positive slip (or negative slip) if $a_{12} > 0$ (or $a_{12} < 0$). Furthermore, the tangential compliance has no effect on the condition of uniqueness and existence of solutions for Signorini contact with Coulomb friction, but the domains of stick and slip solutions are changed.

Secondly, if $k_a \neq 0$ and $C_T \to 0$, then we get from eqn (51) that:

$$\Lambda_A = a_{11} - \mu a_{12}\,\mathrm{sgn}\,(P_T) + \frac{k_a P_N}{3P_s}[a_{12}\,\mathrm{sgn}\,(P_T) - \mu a_{22}],$$

which gives the condition on uniqueness and existence of solutions for Signorini contact with Coulomb friction and Archard's law of wear. Compared to Λ_C, Λ_A also depends on k_a, P_N, P_s and a_{22}. For a sufficiently large contact force, non-uniqueness and non-existence can appear for both positive and negative slip, and not only for positive slip or negative slip as in the previous case, depending on the sign of a_{12}.

Finally, if $k_a \neq 0$ and $C_T \neq 0$, then the range of uniqueness and existence of solutions is decreased further compared to the two previous cases, because $\det[A] = a_{11}a_{22} - a_{12}^2 > 0$ in eqn (51).

5. CONCLUDING REMARKS

In Section 2 a generalized standard model for fretting is derived from the principle of virtual power, the balance of energy and the second law of thermodynamics. The model is defined by a free energy and a dual pseudo-potential. A certain internal state variable is introduced to model the wear process at the interface. It is interpreted as a normal gap between the bodies owing to wear. In a similar way other internal state variables may be defined to model other interfacial phenomena.

One may notice that it is necessary to treat the thermal model as a three-body model to include heat transfer across the contact interface. Otherwise, the temperatures of the contact surfaces must be equal. In the authors' opinion, an important extension of the model presented in this paper should be to formulate a complete three-body model, within the framework of continuum thermodynamics, which includes both thermal and mechanical effects.

In Section 3 some specific forms of the generalized standard model are suggested. For instance, a free energy corresponding to an extension of Signorini's unilateral contact conditions accounting for wear processes at the interface and having a linear tangential compliance between the tangential displacement and the tangential contact traction is suggested. Moreover, a dual pseudo-potential with a friction and wear limit criterion equivalent to Coulomb's law of friction and Archard's law of wear is given. Other friction and wear criteria are also discussed.

The extension of Signorini's unilateral contact conditions is obtained by use of the internal state variable defined for the wear process. A contact law with normal compliance can be extended in a similar way. This type of extensions belongs to a class of free energies where the wear driving force is equal to the contact pressure. Specific forms of other classes of free energies have not been considered in this work.

In Section 4 the specific free energy and dual pseudo-potential mentioned above are analysed for a one point elastic contact problem, where the so-called rate problem is solved. In the intermediate contact state, i.e. when both the contact force and the gap are equal to zero, it is shown that solutions exist for all loading directions, but they may be non-unique. The uniqueness depends on μ, a_{11} and a_{12}. Moreover, no wear is developed in the intermediate contact state. Wear is only developed in the contact state with positive or negative slip and the contact pressure greater than zero. In this contact state, it is seen that non-uniqueness and non-existence of solutions may appear, depending on μ, a_{11}, a_{12}, a_{22}, C_T, k_a, P_s and P_N. In the other contact states there always exists a unique solution for all loading directions.

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APPENDIX A. BASIC CONVEX ANALYSIS

Definitions

1. A function $f: \mathscr{R}^n \to \mathscr{R} \cup \{+\infty\}$, not identical to $+\infty$, is said to be *convex* when, for all $(x, x') \in \mathscr{R}^n \times \mathscr{R}^n$ and all $\alpha \in [0, 1]$, there holds:

$$f(\alpha x + (1-\alpha)x') \leqslant \alpha f(x) + (1-\alpha) f(x').$$

2. A set $K \subset \mathscr{R}^n$ is said to be *convex* if $\alpha x + (1-\alpha)x' \in K$ whenever $(x, x') \in K \times K$ and $\alpha \in [0, 1]$.

3. The *indicator function* $I_K: \mathscr{R}^n \to \mathscr{R} \cup \{+\infty\}$ for a non-empty set $K \subset \mathscr{R}^n$ is defined by:

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{otherwise.} \end{cases}$$

4. The *subdifferential* of f at x is the set defined by:

$$\partial f(x) = \{\mathscr{X} : f(x') \geqslant f(x) + \langle \mathscr{X}, x'-x \rangle \forall x' \in \mathscr{R}^n\}, \tag{52}$$

where $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathscr{R}^n$.

5. The *normal cone* N_K of a closed convex set K is defined by:

$$N_K(x) = \{\mathscr{X} : \langle \mathscr{X}, x'-x \rangle \leqslant 0 \,\forall x' \in K\}.$$

Propositions

1. If K is closed and convex, then:

$$\partial I_K(x) = \begin{cases} N_K(x) & \text{if } x \in K \\ \emptyset & \text{otherwise.} \end{cases}$$

2. If:

$$K = \{x : g(x) \leqslant 0\},$$

with g given as a convex differentiable function, and some constraints qualifications are satisfied, then an element of $N_K(x)$ can be expressed as:

$$\mathscr{X} = \lambda \nabla g, \quad \lambda \geqslant 0, \quad g(x) \leqslant 0, \quad \lambda g(x) = 0.$$

3. If $\mathscr{X} \in \partial f(x)$, $0 = f(0)$ and $0 \in \partial f(0)$ then:

$$\langle \mathscr{X}, x \rangle \geqslant 0$$

for all $x \in \mathscr{R}^n$. This can be seen from eqn (52) by first taking $x' = 0$ and $x = x$:

$$0 = f(0) \geqslant f(x) + \langle \mathscr{X}, 0-x \rangle$$

and then taking $x' = x$ and $x = 0$:

$$f(x) \geqslant 0 + \langle 0, x-0 \rangle = 0.$$

This together forms:

$$\langle \mathscr{X}, x \rangle \geqslant f(x) \geqslant 0,$$

which implies the above statement.

For a full presentation of convex analysis see Hiriart-Urruty and Lemaréchal (1993).

APPENDIX B. TAYLOR EXPANSIONS OF THE STATE LAWS AND THE COMPLEMENTARY LAWS

The rate laws in Section 4.2 are derived by making a Taylor expansion in a time increment Δt of the state laws in eqn (48) and the complementary laws in eqn (49), and then for each particular contact state letting Δt approach zero. The Taylor expansions of interest are presented in this appendix.

The Taylor expansions of eqn (48) are:

$$P_N(t+\Delta t) = P_N(t) + \Delta t \dot{P}_N(t) + \mathcal{O}(\Delta t^2) \geqslant 0,$$

$$w_N(t+\Delta t) - w^w(t+\Delta t) = w_N(t) - w^w(t) + \Delta t[\dot{w}_N(t) - \dot{w}^w(t)] + \mathcal{O}(\Delta t^2) \leqslant 0,$$

$$\begin{aligned} P_N(t+\Delta t)[w_N(t+\Delta t) - w^w(t+\Delta t)] &= P_N(t)[w_N(t) - w^w(t)] \\ &+ \Delta t\{P_N(t)[\dot{w}_N(t) - \dot{w}^w(t)] + \dot{P}_N(t)[w_N(t) - w^w(t)]\} + \Delta t^2\{\tfrac{1}{2}P_N(t)[\ddot{w}_N(t) - \ddot{w}^w(t)] \\ &+ \tfrac{1}{2}\ddot{P}_N(t)[w_N(t) - w^w(t)] + \dot{P}_N(t)[\dot{w}_N(t) - \dot{w}^w(t)]\} + \mathcal{O}(\Delta t^3) = 0, \end{aligned}$$

$$w_T^r(t+\Delta t) = w_T^r(t)+\Delta t\dot{w}_T^r(t)+\mathcal{O}(\Delta t^2) = C_T P_T(t)+\Delta t C_T \dot{P}_T(t)+\mathcal{O}(\Delta t^2).$$

The Taylor expansions of eqn (49) are:

$$\dot{w}_T^i(t+\Delta t) = \dot{w}_T^i(t)+\mathcal{O}(\Delta t) = [\dot{\lambda}(t)+\mathcal{O}(\Delta t)]\frac{P_T(t)+\Delta t\dot{P}_T(t)+\mathcal{O}(\Delta t^2)}{|P_T(t)+\Delta t\dot{P}_T(t)+\mathcal{O}(\Delta t^2)|},$$

$$|P_T(t+\Delta t)|-\mu P_N(t+\Delta t) = |P_T(t)|-\mu P_N(t)+\Delta t\{\operatorname{sgn}[P_T(t)]\dot{P}_T(t)-\mu\dot{P}_N(t)\}+\mathcal{O}(\Delta t^2) \leqslant 0,$$

$$\dot{\lambda}(t+\Delta t)[|P_T(t+\Delta t)|-\mu P_N(t+\Delta t)] = \dot{\lambda}(t)[|P_T(t)|-\mu P_N(t)] + \Delta t(\dot{\lambda}(t)\{\operatorname{sgn}[P_T(t)]\dot{P}_T(t)-\mu\dot{P}_N(t)\}+\ddot{\lambda}(t)[|P_T(t)|-\mu P_N(t)])+\mathcal{O}(\Delta t^2) = 0.$$

If $P_T(t) = 0$, then $\operatorname{sgn}[P_T(t)]\dot{P}_T(t)$ is changed to $|\dot{P}_T(t)|$ in the expressions above.